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Bound-state asymptotic estimates for window-coupled Dirichlet strips and layers

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Abstract. We consider the discrete spectrum of the Dirichlet Laplacian on a manifold consisting of two adjacent parallel straight strips or planar layers coupled by a finite number N of windows in the common boundary. If the windows are small enough, there is just one isolated eigenvalue. We find upper and lower asymptotic bounds on the gap between the eigenvalue and the essential spectrum in the planar case, as well as for $N = 1$ in three dimensions. Based on these results, we formulate a conjecture on the weak-coupling asymptotic behaviour of such bound states.

1. Introduction

There has recently been some interest in Laplacians on strips or layers. Such a system is trivial when the manifold is straight and the boundary conditions are translation-invariant, so there is a natural separation of variables. On the other hand, the spectral properties become non-trivial if the transverse modes are coupled, which can be achieved, e.g., if the manifold is bent, locally deformed, or coupled to another one [EŠ, DE, BGRS, EŠTV, EV1, EV2].

The interest stems from two sources. On the physical side, such operators with Dirichlet boundary conditions are used as models of various mesoscopic semiconductor structures. The corresponding solid-state literature is rather rich, see [DE, EŠTV] for some references. On the other hand, bound states in systems with open geometries also pose mathematical questions such as the weak-coupling limit, validity of the semiclassical approximation, resonance scattering in such structures, etc. Some of their properties can be seen numerically [EŠTV] while analytical proofs are lacking. Recall also that a closely related problem concerns Neumann Laplacians, namely the existence of trapped modes in acoustic waveguides [ELV, DE].

In a recent paper [EV1] we studied a pair of parallel Dirichlet strips of widths d_1, d_2 coupled laterally through a window of a width $2a$ in the common boundary; we have shown that there are positive $c_1,\ c_2$ such that the gap between the ground state and the threshold of the continuous spectrum can be estimated as

$$-c_1 a^4 \leqslant \epsilon(a) - \left(\frac{\pi}{d}\right)^2 \leqslant -c_2 a^4 \tag{1.1}$$

for any a sufficiently small. The numerical result of [EŠTV] suggests that the true asymptotics are of the same type, but proving this assertion and finding the coefficient in the leading term remains an open problem.

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The aim of the present paper is to generalize the above inequalities to the case of a finite number of connecting windows and to a higher dimension. In section 2 we shall prove the bounds for a pair of strips with N windows. In section 3 we formulate the analogous problem for two layers and prove two-sided asymptotic bounds for a single window shrinking to a point. The proofs rely in both cases on variational estimates and follow the same basic strategy as in [EV1]. On the other hand, the existence of multiple windows or the change in dimension require numerous modifications, which prompts us to present the argument in sufficient detail.

The upper and lower asymptotics bounds we will derive are of the same type in each case, differing just by values of the constants. We are convinced that ground state has an asymptotic expansion and its lowest-order is given by functions analogous to our bounds. This conjecture is formulated in section 4. At the same time, our present method does not allow us to squeeze the bounds, or even to come close to the true values, as remark 2.2 below illustrates.

2. N windows in dimension two

Consider a straight planar strip $\Sigma := \mathbb{R} \times [-d_2, d_1]$. Given finite sequences $\mathcal{C} \equiv \{x_k\}_{k=1}^N$ of mutually distinct points of the x-axis and $A = \{a_k\}_{k=1}^N$ with $a_k > 0$, we denote $\mathcal{W}_k := [x_k - a_k, x_k + a_k]$ and set $\mathcal{W} := \bigcup_{k=1}^n \mathcal{W}_k$. Then we define $H(d_1, d_2; \mathcal{W})$ as the Laplacian on $L^2(\Sigma)$ subject to the Dirichlet condition at $y = -d_2, d_1$ as well as at the $\mathbb{R}\setminus\mathcal{W}$ part of the x-axis; this operator coincides with the Dirichlet Laplacian at the strip with the appropriate piecewise cut (see figure 1) defined in the standard way [RS4, section XIII.15]. Following the notation introduced in [EV1] we put $d := \max\{d_1, d_2\}$ and $D := d_1 + d_2$. If $d_1 = d_2$, the operator decomposes into an orthogonal sum with respect to the y-parity; the non-trivial part is unitarily equivalent to the Laplacian on $L^2(\Sigma_+)$, where $\Sigma_+ := \mathbb{R} \times [0, d]$, with the Neumann condition at window part $\mathcal{W}$ of the x-axis and Dirichlet at the remaining part of the boundary; we denote it by $H(d; \mathcal{W})$. If the specification is clear from the context, we will often denote the operator in question simply as H.

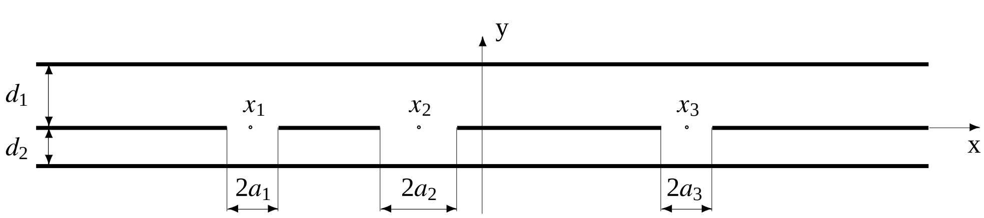

Figure 1. Window-coupled planar waveguides.

We need a quantity to express the 'smallness' of the window set. We define

$$I(\mathcal{W}) := \sum_{k=1}^{N} a_k |\mathcal{W}_k| = 2 \sum_{k=1}^{N} a_k^2. \tag{2.1}$$

Then the result of [EV1] generalizes to the present situation as follows.

Theorem 2.1. $\sigma_{\mathrm{ess}}(H(d_1, d_2; \mathcal{W})) = [(\pi/d)^2, \infty)$. The discrete spectrum is contained in $((\pi/D)^2, (\pi/d)^2)$, finite, and non-empty provided $\mathcal{W} \neq \emptyset$. If $I(\mathcal{W})$ is sufficiently small,

$\sigma_{\mathrm{disc}}(H(d_1, d_2; \mathcal{W}))$ consists of just one simple eigenvalue $\epsilon(\mathcal{W}) < (\pi/d)^2$ and there are positive c_1, c_2 such that

$$-c_1 I(\mathcal{W})^2 \leqslant \epsilon(a) - \left(\frac{\pi}{d}\right)^2 \leqslant -c_2 I(\mathcal{W})^2 \tag{2.2}$$

holds for any $I(\mathcal{W})$ sufficiently small.

Proof. (a) The upper bound. In the symmetric case, $d_1 = d_2$, the trial function will be chosen as $\psi = F + G$, where

$$F(x, y) := f_1(x)\chi_1(y) \tag{2.3}$$

with

$$f_1(x) := \max\left\{\chi_{[x_1-a_1, x_N+a_N]}(x),\ \mathrm{e}^{-\kappa|x-x_1+a_1|},\ \mathrm{e}^{-\kappa|x-x_N-a_N|}\right\}$$

and

$$G(x, y) := \sum_{k=1}^{N} G_k(x, y) \tag{2.4}$$

with

$$G_k(x, y) := \frac{2\eta_k a_k}{|\mathcal{W}|}\chi_{[x_k-a_k, x_k+a_k]}(x)\cos\left(\frac{\pi(x-x_k)}{2a_k}\right)R_k(y) \tag{2.5}$$

where $|\mathcal{W}| := 2\sum_{k=1}^{N} a_k$, and

$$R_k(y) := \begin{cases} \mathrm{e}^{-\pi y/2a_k} & y \in [0, \frac{1}{2}d] \\ 2(1-y/d)\mathrm{e}^{-\pi d/4a_k} & y \in [\frac{1}{2}d, d] \end{cases} \tag{2.6}$$

for $k = 1, 2, \ldots, N$. As before $\chi_n(y) = \sqrt{2/d}\sin(\pi n y/d)$, $n = 1, 2, \ldots$, denote the 'transverse' eigenfunctions—not to be confused with the indicator function χ_M of a set M. Note that as long as we work with trial functions of $Q(H)$, the window smoothing employed in [EV1] is not needed (cf [RS4]).

The functional $L(\psi) := (H\psi, \psi) - (\pi/d)^2\|\psi\|^2$ can be expressed as

$$L(\psi) = \|\psi_x\|^2 + \|G_y\|^2 - \left(\frac{\pi}{d}\right)^2\|G\|^2 - 2\frac{\pi}{d}\sqrt{\frac{2}{d}}\sum_{k=1}^{N}\int_{x_k-a_k}^{x_k+a_k} G_k(x, 0)\,\mathrm{d}x. \tag{2.7}$$

Since f_x, G_x have disjoint supports, we have $\|\psi_x\|^2 = \|F_x\|^2 + \sum_{k=1}^{N}\|G_{k,x}\|^2$, where $G_{k,x} := \partial_x G_k$. The kth term of the last sum equals $\eta_k^2\pi^2 a_k|\mathcal{W}|^{-2}\|R_k\|^2_{L^2(0,d)}$, and

$$\|R_k\|^2_{L^2(0,d)} = \frac{a_k}{\pi} + \left(\frac{d}{6} - \frac{a_k}{\pi}\right)\mathrm{e}^{-\pi d/2a_k} < \frac{a_k}{\pi}(1+\varepsilon_1)$$

for any $\varepsilon_1 > 0$ and a_k sufficiently small. Obviously, $\int_{x_k-a_k}^{x_k+a_k} G_k(x, 0)\,\mathrm{d}x = (8/\pi)\eta_k a_k^2|\mathcal{W}|^{-1}$, and furthermore, a bound on $\|G_{k,y}\|^2$ follows from

$$\|R_k'\|^2_{L^2(0,d)} = \frac{\pi}{4a_k} + \left(\frac{2}{d} - \frac{\pi}{4a_k}\right)\mathrm{e}^{-\pi d/2a_k} < \frac{\pi}{4a_k}$$

for $a_k < \pi d/8$, which means that $\|G_{k,y}\|^2 < \pi\sum_k \eta_k^2 a_k^2|\mathcal{W}|^{-2}$. Now we can put these estimates together using $\|F_x\|^2 = \kappa$; neglecting the negative term $-(\pi/d)^2\|G\|^2$, we arrive at the inequality

$$L(\psi) < \kappa - \frac{16\sqrt{2}}{d^{3/2}}\sum_{k=1}^{N}\frac{\eta_k a_k^2}{|\mathcal{W}|} + \pi(2+\varepsilon_1)\sum_{k=1}^{N}\frac{\eta_k^2 a_k^2}{|\mathcal{W}|^2}.$$

The sum of the last two terms on the right-hand side is minimized by $-\big(2^7/\pi d^3(2+\varepsilon_1)\big)\sum_k a_k^2$. To conclude the argument, we have to estimate the trial function norm $\|\psi\|^2$ from below. The tail part is $\|\psi\|^2_{x\in\mathbb{R}\setminus\mathcal{W}} = \kappa^{-1}$, while the window contributes by

$$\|\psi\|^2_{x\in\mathcal{W}} \leqslant 2\|F\|^2_{x\in\mathcal{W}} + 2\|G\|^2_{x\in\mathcal{W}} = |x_N - x_1 + a_N + a_1| + 4\sum_{k=1}^N \frac{\eta_k^2 a_k^3}{|\mathcal{W}|^2}\|R_k\|^2_{L^2(0,d)}$$

so $\|\psi\|^2 > (1-\varepsilon_2)\kappa^{-1}$ holds for any $\varepsilon_2 > 0$ provided $|\mathcal{W}|$ is sufficiently small. Minimizing the above-obtained estimate of $L(\psi)/\|\psi\|^2$ over κ, we find

$$\frac{L(\psi)}{\|\psi\|^2} < -(1-\varepsilon_2)^{-1}\left(\frac{2^6}{\pi d^3(2+\varepsilon_1)}\sum_{k=1}^N a_k^2\right)^2 \tag{2.8}$$

which yields the upper bound in (2.2) for $d_1 = d_2$. The extension to the non-symmetric case proceeds as for $N = 1$; the trial function is chosen in the above form for the wider channel, while in the narrower one it is given by (2.4) rescaled transversely.

Remark 2.2. The bound can be improved, for instance, by replacing the factorized form (2.5) by a series, whose terms will be products of the trigonometric basis in the window with the functions $R_{k,n}(y)$ decaying as $\exp\{-\pi n y/2a_k\}$ about $y = 0$ (in the above estimate we used just the first term of such a series). However, the gain is not large. To illustrate this fact, take $N = 1$ and $d = \pi$. The use of the series leads then to the upper bound $(2a/\pi)^4$, improving the coefficient by $(\pi^2/8)^2 \approx 1.52$. A comparison with the numerically determined ground state [EŠTV] shows that the true asymptotic behaviour should be $\approx (2.23\,a)^4$, so the c_2 obtained is still two orders of magnitude wide of the mark. The reason is obviously that the wavefunction is affected by the window outside the transverse 'window strip' as well.

Before proceeding to the lower bound, let us state some auxiliary results.

Lemma 2.3. Let $J[\phi] := \int_a^b \big(\phi'(t)^2 + m^2\phi(t)^2\big)\,\mathrm{d}t$ for $\phi \in C^2(a,b)$ with $\phi(a) = c_a$ (a fixed number). Given $m_0 > 0$, there is $\alpha_0 > 0$ such that

$$J[\phi] \geqslant \alpha_0 m c_a^2 \tag{2.9}$$

holds for all $m \geqslant m_0$.

Proof. The mimimum is obviously reached with $\phi'(b) = 0$. The corresponding Euler's equation is solved by $\phi_0(t) = d_1\mathrm{e}^{-mt} + d_2\mathrm{e}^{mt}$, where $d_1 = c_a(\mathrm{e}^{-ma} + \mathrm{e}^{m(a-2b)})^{-1}$ and $d_2 = d_1\mathrm{e}^{-2mb}$. Since $m^{-1}c_a^{-2}\inf J(\phi) > 0$ for any $m \geqslant m_0$, it is sufficient to check that (2.9) remains valid as $m \to \infty$; evaluating the functional for ϕ_0 we find $\lim_{m\to\infty} J(\phi) = mc_a^2$. □

Lemma 2.4. Suppose that ϕ minimizes $J[\phi] := \int_a^{2a}\big(\phi'(t)^2 + p^2\phi(t)^2\big)\,\mathrm{d}t$ for positive a, p within $C^2(a,2a)$ with the boundary condition $\phi(a) = c_a$; then

$$|\phi(2a)| \leqslant 2|c_a|\mathrm{e}^{-pa}. \tag{2.10}$$

Proof. Assume for definiteness that $c_a > 0$. Again by the symmetry argument already mentioned, $\phi'(2a) = 0$, and its explicit form is $\phi(t) = c_a\cosh p(2a-t)/\cosh pa$, which yields $\phi(2a) \leqslant 2c_a\mathrm{e}^{-pa}$. □

For the sake of completeness we also reproduce the following assertion, the proof of which is given in [EV1].

Lemma 2.5. Let $\phi \in C^2[0,d]$ with $\phi(0)=\beta$ and $\phi(d)=0$. If $(\phi,\chi_1)=0$, then for every $m>0$ there is $d_0>0$ such that

$$\int_0^d \phi'(t)^2\,\mathrm{d}t + \left(\frac{m}{a}\right)^2 \int_0^a \phi(t)^2\,\mathrm{d}t - \left(\frac{\pi}{d}\right)^2 \int_0^d \phi(t)^2\,\mathrm{d}t \geqslant \frac{d_0\beta^2}{a} \tag{2.11}$$

holds for all a sufficiently small.

(b) Proof of theorem 2.1, continued. The lower bound is again the more difficult of the two; however, we may restrict ourselves to the symmetric case alone, because by inserting an additional Neumann boundary into the window we get a lower bound, and therefore in what follows we consider the spectrum of $H \equiv H(d;\mathcal{W})$.

We begin with a simple observation that it is sufficient to estimate $L(\psi) := (H\psi,\psi) - (\pi/d)^2\|\psi\|^2$ from below for all *real* ψ of a core of H, say, all C^2-smooth $\psi \in L^2(\Sigma_+)$ satisfying the boundary conditions, since H commutes with complex conjugation. The main difficulty caused by the existence of multiple windows is that we are no longer allowed to restrict ourselves to trial functions symmetric with respect to the window centres. The strategy we employ is at the start to split off a part of the kinetic-energy contribution to the functional, say, $\frac{1}{4}\|\psi_x\|^2$, which at the end will be used to mend the problems coming from the asymmetry, i.e. we begin by estimating $L_0(\psi) := L(\psi) - \frac{1}{4}\|\psi_x\|^2$.

A trial function of the indicated set will be written in the form of a Fourier series

$$\psi(x,y) = \sum_{n=1}^{\infty} c_n(x)\chi_n(y) \tag{2.12}$$

with smooth coefficients $c_n(x) = (\psi(x,\cdot),\chi_n)$, which is uniformly convergent outside the windows, $x \notin \mathcal{W}$. We split the lowest transverse-mode coefficient further by setting

$$f_1 := c_1 - \sum_{k=1}^{N} \hat{f}_k \tag{2.13}$$

where

$$\hat{f}_k := \begin{cases} c_k(x) - \alpha_k & x \in [x_k - 2a_k, x_k + a_k] \\ 0 & \text{otherwise} \end{cases} \tag{2.14}$$

with $\alpha_k := c_1(x_k - 2a_k)$, i.e. each one of the functions $\hat{f}_k$ vanishes at the left end-point of the appropriate extended window; in contrast to [EV1] we double only the left half of the window. Writing the full trial function as

$$\psi(x,y) = F(x,y) + G(x,y) \qquad F(x,y) := f_1(x)\chi_1(y) \tag{2.15}$$

we can cast the reduced energy functional in the form

$$L_0(\psi) = \frac{3}{4}\|\psi_x\|^2 + \|G_y\|^2 - \left(\frac{\pi}{d}\right)^2\|G\|^2 - \sum_{k=1}^{N} 2\alpha_k \frac{\pi}{d}\sqrt{\frac{2}{d}} \int_{\mathcal{W}_k} G(x,0)\,\mathrm{d}x. \tag{2.16}$$

Contributions to (2.16) from different parts of the strip Σ_+ will be estimated separately. The out-of-window part consists of the sets

$$\omega_1 = \{(x,y) :\ x \leqslant x_1 - a_1\}$$

$$\omega_k = \{(x,y) :\ x_{k-1} + a_{k-1} \leqslant x \leqslant x_k - a_k\} \qquad k = 2,\ldots,N$$

$$\omega_{N+1} = \{(x,y) :\ x \geqslant x_N + a_N\}.$$

The expansion (2.12) yields

$$\frac{1}{4}\|\psi_x\|^2_{\omega_k} + \|G_y\|^2_{\omega_k} - \left(\frac{\pi}{d}\right)^2 \|G\|^2_{\omega_k}$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\int_{\omega_k} c_n'(x)^2\,\mathrm{d}x + \sum_{n=1}^{\infty}\left(\frac{\pi}{d}\right)^2 (n^2-1)\int_{\omega_k} c_n(x)^2\,\mathrm{d}x$$

and therefore

$$\frac{1}{4}\|\psi_x\|^2_{\omega_k} + \|G_y\|^2_{\omega_k} - \left(\frac{\pi}{d}\right)^2 \|G\|^2_{\omega_k} > \mu_0 \sum_{n=2}^{\infty} n c_n (x_k - a_k)^2$$

with some $\mu_0 > 0$ follows from lemma 2.3 (for $a = x_k - a_k$ and $k = 2, \ldots, N$). The same inequality for $k = 1$ is derived as in [EV1]; for the right tail we use just the fact that the expression is positive so we can neglect it. Since $\psi_x = G_x$ inside the (left extended) windows, we arrive at the bound

$$L_0(\psi) > \frac{1}{2}\|\psi_x\|^2_{x\notin\mathcal{W}} + \sum_{k=1}^{N}\left\{\frac{3}{4}\|G_x\|^2_{x\in\mathcal{W}_k} + \|G_y\|^2_{x\in\mathcal{W}_k} - \left(\frac{\pi}{d}\right)^2\|G\|^2_{x\in\mathcal{W}_k}\right.$$
$$\left. +\mu_0\sum_{n=2}^{\infty} n c_n (x_k - a_k)^2 - 2\alpha_k \frac{\pi}{d}\sqrt{\frac{2}{d}}\int_{\mathcal{W}_k} G(x,0)\,\mathrm{d}x\right\}. \tag{2.17}$$

Our next goal is to estimate the contribution to $\|G_x\|^2$ from the extended windows, $\mathcal{E}_k := [x_k - 2a_k, x_k + a_k]$. In contrast to the case $N = 1$, however, even the lowest-mode projection of G may not vanish at the right end-points of these intervals, so the inequality [EV1, equation (5.6)] has to be modified. Fortunately, it is sufficient to change the coefficient: if a function $\tilde{G} : \Sigma_+ \to C^2(\Sigma_+)$ vanishes for $x = x_k - 2a_k$, the inequality [EV1, equation (4.2)] in combination with a symmetry argument imply

$$\|\tilde{G}_x\|^2_{x\in\mathcal{E}_k} \geqslant \left(\frac{\pi}{6a_k}\right)^2 \|\tilde{G}\|^2_{x\in\mathcal{E}_k}. \tag{2.18}$$

To use this result we split the function by singling out the projection of G onto the first transverse mode

$$G(x,y) = G_1(x,y) + G_2(x,y) \qquad G_1(x,y) = \sum_{k=1}^{N} \hat{f}_k(x)\chi_1(y). \tag{2.19}$$

We have

$$\tfrac{1}{2}\|\psi_x\|^2_{x\in\mathcal{E}_k\setminus\mathcal{W}_k} + \tfrac{3}{4}\|G_x\|^2_{x\in\mathcal{W}_k} \geqslant \tfrac{1}{2}\|G_x\|^2_{x\in\mathcal{E}_k} = \tfrac{1}{2}\|G_{1,x}\|^2_{x\in\mathcal{E}_k} + \tfrac{1}{2}\|G_{2,x}\|^2_{x\in\mathcal{E}_k}$$

and therefore

$$L_0(\psi) > \frac{1}{2}\|\psi_x\|^2_{x\notin\mathcal{E}} + \sum_{k=1}^{N}\left\{\frac{1}{2}\|G_{2,x}\|^2_{x\in\mathcal{E}_k} + \|G_y\|^2_{x\in\mathcal{W}_k} - \left(\frac{\pi}{d}\right)^2\|G\|^2_{x\in\mathcal{W}_k}\right.$$
$$+\mu_0\sum_{n=2}^{\infty} n c_n (x_k - a_k)^2 - 2\alpha_k \frac{\pi}{d}\sqrt{\frac{2}{d}}\int_{\mathcal{W}_k} G(x,0)\,\mathrm{d}x$$
$$\left. +\frac{1}{2}\left(\frac{\pi}{6a_k}\right)^2 \|G_1\|^2_{x\in\mathcal{E}_k}\right\} \tag{2.20}$$

with $\mathcal{E} := \bigcup_{k=1}^N \mathcal{E}_k$. To proceed further we split the function G_2 in the kth extended window as $G_2(x, y) = \hat{G}(x, y) + \Gamma(x, y)$, where

$$\Gamma(x, y) := \sum_{n=2}^{\infty} c_n(x_k - 2a_k)\chi_n(y).$$

The second part is independent of x, while the first vanishes at the left end-point, so $G_{2,x} = \hat{G}_x$ may be estimated by means of (2.18) and the Schwarz inequality as

$$\begin{aligned} \|G_{2,x}\|^2_{x\in\mathcal{E}_k} &\geqslant \left(\frac{\pi}{6a_k}\right)^2 \|\hat{G}\|^2_{x\in\mathcal{E}_k} \geqslant \left(\frac{\pi}{6a_k}\right)^2 \|\hat{G}\|^2_{\Omega_k} \\ &\geqslant \frac{1}{2}\left(\frac{\pi}{6a_k}\right)^2 \|G_2\|^2_{\Omega_k} - \left(\frac{\pi}{6a_k}\right)^2 \|\Gamma\|^2_{\Omega_k} \end{aligned} \tag{2.21}$$

where we have denoted $\Omega_k := \mathcal{E}_k \times [0, a_k]$. To make use of the last estimate we have to find an upper bound on $\|\Gamma\|^2_{\Omega_k}$. To this end we note the following.

(i) Instead of assuming $c_n \in C^2$, the lower bound can be sought in a wider class of ψ with piecewise continuous coefficients.

(ii) On the other hand, we may restrict ourselves to those ψ which satisfy for $x \in \mathcal{E}_k\setminus\mathcal{W}_k$ and $n \geqslant 2$ the inequality

$$|c_n(x)| \leqslant c_n^{\mathrm{ex}}(x) := |c_n(a)| \frac{\cosh\big((\pi/d)\sqrt{n^2-1}(x - x_k + 2a_k)\big)}{\cosh\big((\pi a_k/d)\sqrt{n^2-1}\big)}. \tag{2.22}$$

To see this we split the trial function in analogy with [EV1]:

$$\tilde{\psi}(x, y) := \begin{cases} \psi(x, y) - c_n(x)\chi_n(y) & x \in \mathcal{E}_k \setminus \mathcal{W}_k \\ \psi(x, y) & \text{otherwise.} \end{cases}$$

The basic expression $L(\psi)/\|\psi\|^2$ can be then rewritten as

$$\frac{\tilde{L}(\tilde{\psi}) - (\pi/d)^2\|\tilde{\psi}\|^2 + \sum_{k=1}^N \int_{\mathcal{E}_k\setminus\mathcal{W}_k} \big[c_n'(x)^2\,\mathrm{d}x + \big((\pi/d)\sqrt{n^2-1}\big)^2 c_n(x)^2\big]\,\mathrm{d}x}{\|\tilde{\psi}\|^2 + \sum_{k=1}^N \int_{\mathcal{E}_k\setminus\mathcal{W}_k} c_n(x)^2\,\mathrm{d}x}$$

where $\tilde{L}(\tilde{\psi}) := \int_{\Sigma_+} \big(|\tilde{\psi}_x|^2 + |\tilde{\psi}_y|^2\big)(x, y)\,\mathrm{d}x\,\mathrm{d}y$. We may assume only those ψ for which the numerator is negative; the part of its last term corresponding to the 'window neighbourhoods' is minimized by the hyperbolic function c_n^{ex} of (2.22) (see the proof of lemma 2.4). It follows that replacing $c_n(x)^2$ by $\min\{c_n(x)^2, c_n^{\mathrm{ex}}(x)^2\}$ we can only get a larger negative number, while the positive denominator can only be diminished.

To estimate the norm of Γ restricted to Ω_k, we adapt again the argument of [EV1] and divide the series into parts referring to small and large values of y, and respectively employ the smallness of $\|\chi_n \upharpoonright [0, a]\|$ and the bound (2.22). This yields

$$\begin{aligned} \|\Gamma\|^2_{\Omega_k} &= \int_{\mathcal{E}_k} \mathrm{d}x \int_0^{a_k} \mathrm{d}y \left(\sum_{n=2}^{\infty} c_n[2a_k]\chi_n(y)\right)^2 \\ &\leqslant 6a_k \int_0^{a_k} \left(\sum_{n=2}^{[a_k^{-1}]+1} c_n[2a_k]\chi_n(y)\right)^2 \mathrm{d}y + \end{aligned}$$

$$+\,6a_k\int_0^{a_k}\Bigg(\sum_{2\leqslant n=[a_k^{-1}]+2}^{\infty} c_n[2a_k]\chi_n(y)\Bigg)^2\,\mathrm{d}y$$

$$\leqslant 24a_k\Bigg(\sum_{n=2}^{[a_k^{-1}]+1} n^{-1}c_n[a_k]^2\int_0^{a_k}\chi_n(y)^2\,\mathrm{d}y\Bigg)\Bigg(\sum_{n=2}^{[a_k^{-1}]+1} n\Bigg)$$

$$+\,24a_k\Bigg(\sum_{2\leqslant n=[a_k^{-1}]+2}^{\infty} nc_n[a_k]^2\int_0^{a_k}\chi_n(y)^2\,\mathrm{d}y\Bigg)$$

$$\times\Bigg(\sum_{2\leqslant n=[a_k^{-1}]+2}^{\infty} n^{-1}\mathrm{e}^{-(2\pi a_k/d)\sqrt{n^2-1}}\Bigg)$$

where $c_n[ja_k] := c_n(x_k - ja_k)$ and $[\cdot]$ denotes the entire part; in the the last step we have used the bound $|c_n[2a_k]| < 2|c_n[a_k]|\exp\{-(\pi a_k/d)\sqrt{n^2-1}\}$ which follows from lemma 2.4. In analogy with [EV1], this implies the existence of a positive C_k such that

$$\|\Gamma\|^2_{\Omega_k} \leqslant C_k a_k^2\sum_{n=2}^{\infty} nc_n(x_k - a_k)^2. \tag{2.23}$$

From now on we again consider continuous coefficient functions. By equation (2.21) we have

$$\frac{1}{2}\|G_{2,x}\|^2_{x\in\mathcal{E}_k} + \mu_0\sum_{n=2}^{\infty} nc_n(x_k - a_k)^2$$

$$\geqslant \delta\left(\frac{\pi}{12a_k}\right)^2\|G_2\|^2_{\Omega_k} - \frac{\delta}{2}\left(\frac{\pi}{6a_k}\right)^2\|\Gamma\|^2_{\Omega_k} + \mu_0\sum_{n=2}^{\infty} nc_n(x_k - a_k)^2$$

for an arbitrary $\delta \in (0, 1]$; if we choose the latter sufficiently small, the sum of the last two terms is non-negative for each $k = 1, \ldots, N$ due to (2.23), so

$$L_0(\psi) > \frac{1}{2}\|\psi_x\|^2_{x\notin\mathcal{E}} + \sum_{k=1}^{N}\Bigg\{\|G_y\|^2_{x\in\mathcal{W}_k} - \left(\frac{\pi}{d}\right)^2\|G\|^2_{x\in\mathcal{W}_k} + \frac{m^2}{a_k^2}\|G_2\|^2_{\Omega_k}$$

$$- 2\alpha_k\frac{\pi}{d}\sqrt{\frac{2}{d}}\int_{\mathcal{W}_k} G(x,0)\,\mathrm{d}x + \frac{1}{2}\left(\frac{\pi}{6a_k}\right)^2\|G_1\|^2_{x\in\mathcal{E}_k}\Bigg\} \tag{2.24}$$

where we have denoted $m := \frac{1}{12}\pi\sqrt{\delta}$.

Next we express the first term in the curly bracket using the decomposition (2.19), the properties of the transverse base and integration by parts:

$$\|G_y\|^2_{x\in\mathcal{W}_k} = \|G_{1,y}\|^2_{x\in\mathcal{W}_k} + \|G_{2,y}\|^2_{x\in\mathcal{W}_k} - 2\frac{\pi}{d}\sqrt{\frac{2}{d}}\int_{\mathcal{W}_k}\hat{f}_k(x)G(x,0)\,\mathrm{d}x.$$

As in [EV1] we estimate the last term by the Schwarz inequality, substitute in (2.24), neglect $\|G_{1,y}\|^2_{x\in\mathcal{W}_k}$ as well as

$$\frac{\pi^2}{72a_k^2}\|G_1\|^2_{x\in\mathcal{E}_k} - \frac{\pi(\pi+\sqrt{2})}{d^2}\|G_{1,y}\|^2_{x\in\mathcal{W}_k}$$

which is positive for a_k sufficiently small, obtaining

$$L_0(\psi) > \frac{1}{2}\|\psi_x\|^2_{x\notin\mathcal{E}} + \sum_{k=1}^{N}\Bigg\{\|G_{2,y}\|^2_{x\in\mathcal{W}_k} + \frac{m^2}{a_k^2}\|G_2\|^2_{\Omega_k} - \left(\frac{\pi}{d}\right)^2\|G\|^2_{x\in\mathcal{W}_k}$$

$$- \frac{\pi\sqrt{2}}{d}\|G(\cdot,0)\|^2_{x\in\mathcal{W}_k} - 2\alpha_k\frac{\pi}{d}\sqrt{\frac{2}{d}}\int_{\mathcal{W}_k} G(x,0)\,\mathrm{d}x\Bigg\}. \tag{2.25}$$

By lemma 2.5, the sum of the first three terms in the curly bracket is bounded from below by $d_k/a_k\|G(\cdot,0)\|^2_{x\in\mathcal{W}_k}$ for some $d_k > 0$. Since $(d_k/2a_k) - (\pi\sqrt{2}/d) > 0$ holds for a_k sufficiently small, we have

$$L_0(\psi) > \frac{1}{2}\|\psi_x\|^2_{x\notin\mathcal{E}} + \sum_{k=1}^{N}\left\{\frac{d_k}{2a_k}\|G_2(\cdot,0)\|^2_{x\in\mathcal{W}_k} - 4\alpha_k\frac{\pi}{d}\sqrt{\frac{a_k}{d}}\|G_2(\cdot,0)\|_{x\in\mathcal{W}_k}\right\}$$

where we have again employed the Schwarz inequality. The kth term of the sum reaches its minimum with respect to the norm at $-(8\pi^2/d_kd^3)\alpha_k^2a_k^2$. Returning to the original functional and neglecting in the first term of the last estimate all contributions except the one coming from the leftmost component of $\mathbb{R}\setminus\mathcal{E}$, we see that there is a positive γ such that

$$L(\psi) > \frac{1}{4}\|\psi_x\|^2 + \frac{1}{2}\|\psi_x\|^2_{x<x_k-2a_k} - \gamma\sum_{k=1}^{N}\alpha_k^2a_k^2 \tag{2.26}$$

holds provided $|\mathcal{W}|$ is sufficiently small.

To conclude the proof, we denote $\ell_k := x_k - 2a_k - x_1 + 2a_1$ and employ the identity

$$\sum_{k=1}^{N}\alpha_k^2a_k^2 = \sum_{k=1}^{N}\alpha_1^2a_k^2 + \sum_{k=1}^{N}(\alpha_k^2-\alpha_1^2)a_k^2 \tag{2.27}$$

together with the estimate

$$\frac{1}{4}\|\psi_x\|^2 \geqslant \frac{1}{4}\|c_1'\|^2 \geqslant \frac{1}{4N}\sum_{k=1}^{N}\frac{(\alpha_k-\alpha_1)^2}{\ell_k}.$$

If $-\gamma(\alpha_k^2-\alpha_1^2)a_k^2 + (1/4N\ell_k)(\alpha_k-\alpha_1)^2 \geqslant 0$ holds for all $k = 2,\ldots,N$, the bound (2.26) reduces to

$$L(\psi) > \frac{1}{2}\|\psi_x\|^2_{x<x_k-2a_k} - \gamma\alpha_1^2\sum_{k=1}^{N}a_k^2. \tag{2.28}$$

On the other hand, suppose that the end-point values satisfy $\alpha_k - \alpha_1 = \mathcal{O}(a_k)$ as $a_k \to 0$ for $k \in \mathcal{K} \subset \{2,\ldots,N\}$. In view of (2.27) we have

$$L(\psi) > \frac{1}{2}\|\psi_x\|^2_{x<x_k-2a_k} - \gamma\alpha_1^2\sum_{k=1}^{N}a_k^2 + \sum_{k\in\mathcal{K}}\left\{\frac{1}{4N\ell_k}(\alpha_k-\alpha_1)^2 - \gamma(\alpha_k^2-\alpha_1^2)a_k^2\right\}.$$

However, the last term is $\mathcal{O}\left(\sum_{k\in\mathcal{K}}a_k^2\right)$, so equation (2.28) is valid again with a smaller positive coefficient in the last term. Since $\|\psi\|^2 \geqslant 2\int_{-\infty}^{x_1-2a_1}c_1(x)^2\,\mathrm{d}x$, the quantity of interest is bounded from below by

$$\frac{L(\psi)}{\|\psi\|^2} > \frac{\int_{-\infty}^{x_1-2a_1}c_1'(x)^2\,\mathrm{d}x - \gamma\alpha_1^2I(\mathcal{W})}{2\int_{-\infty}^{x_1-2a_1}c_1(x)^2\,\mathrm{d}x}.$$

The right-hand side is minimized by the function $c_1(x) = \alpha_1 e^{\kappa(x-x_1+2a_1)}$, which yields the value $(\kappa^2/2) - \gamma I(\mathcal{W})\kappa$; taking the minimum over κ we find

$$\frac{L(\psi)}{\|\psi\|^2} > -\gamma^2 I(\mathcal{W})^2. \tag{2.29}$$

$\square$

3. Window-coupled layers

The setting of the three-dimensional problem is similar. We have a straight layer, $\Sigma := \mathbb{R}^2 \times [-d_2, d_1]$, and a set $\mathcal{W} \subset \mathbb{R}^2$ which can be written as a finite union, $\mathcal{W} := \cup_{k=1}^N \mathcal{W}_k$, whose components are open, connected sets of non-zero Lebesgue measure; without loss of generality we may suppose they are mutually disjoint. Then we define $H(d_1, d_2; \mathcal{W})$ as the Laplacian on $L^2(\Sigma)$ obeying the Dirichlet condition at the boundary of Σ, i.e. $y = -d_2, d_1$, as well as at $\mathbb{R}^2 \setminus \mathcal{W}$. This operator coincides again with the Dirichlet Laplacian [RS4, section XIII.15] for the sliced layer, the two parts of which are connected through the window set $\mathcal{W}$. We use the same notation as above, $d := \max\{d_1, d_2\}$ and $D := d_1 + d_2$. The non-trivial part of the symmetric case, $d_1 = d_2$, reduces again to analysis of the Laplacian $L^2(\Sigma_+)$, where $\Sigma_+ := \mathbb{R}^2 \times [0, d]$, with the Neumann condition at window part of the plane $y = 0$ and Dirichlet at the remaining part of the boundary; this operator will be denoted as by $H(d; \mathcal{W})$.

Our main aim here is to prove a weak-coupling asymptotic estimate for a pair of layers connected by a single window.

Theorem 3.1. $\sigma_{\mathrm{ess}}(H(d_1, d_2; \mathcal{W})) = [(\pi/d)^2, \infty)$. The discrete spectrum is contained in $((\pi/D)^2, (\pi/d)^2)$, finite, and nonempty provided $\mathcal{W} \neq \emptyset$. Suppose further that $N = 1$ and $\mathcal{W} = aM$ for an non-empty open set M contained in the unit ball $B_1 \subset \mathbb{R}^2$. Then $\sigma_{\mathrm{disc}}(H(d_1, d_2; aM))$ consists of just one simple eigenvalue $\epsilon(aM) < (\pi/d)^2$ for all a sufficiently small, and there are positive c_1, c_2 such that

$$-\exp\left(-c_1 a^{-3}\right) \leqslant \epsilon(a) - \left(\frac{\pi}{d}\right)^2 \leqslant -\exp\left(-c_2 a^{-3}\right). \tag{3.1}$$

Proof. This is again based on variational estimates.

The upper bound in the symmetric case, $d_1 = d_2$, employs the trial function $\psi = F + \eta G$, where $F(x, y) := f_1(x)\chi_1(y)$ again with

$$f_1(x) := \min\left\{1, \frac{K_0(\kappa|x|)}{K_0(\kappa a)}\right\} \tag{3.2}$$

and

$$G(x, y) := \chi_{aM}(x)\phi_1(x)R(y) \tag{3.3}$$

where $\phi_1^{(a)}$ is the ground-state eigenfunction, $\|\phi_1^{(a)}\| = 1$, of the operator $-\Delta_D^{aM}$ corresponding to the positive eigenvalue $\mu_1(a) = \mu_1(1)a^{-2}$, and

$$R(y) := \begin{cases} e^{-\sqrt{\mu_1(a)}y} & y \in [0, \frac{1}{2}d] \\ 2(1 - y/d)\exp\left(-\frac{1}{2}d\sqrt{\mu_1(a)}\right) & y \in [\frac{1}{2}d, d]. \end{cases} \tag{3.4}$$

Using $-\chi_1'' = (\pi/d)^2\chi_1$, a simple integration by parts, and the fact that the vector functions ∇f_1 and $\nabla\phi_1^{(a)}$ have disjoint supports, we can express the reduced energy functional

$L(\psi) := (H\psi, \psi) - (\pi/d)^2\|\psi\|^2$ as

$$L(\psi) = \|\nabla f_1\|^2_{L^2(\mathbb{R}^2)} + \eta^2\left(\mu_1(a) - \left(\frac{\pi}{d}\right)^2\right)\|R\|^2_{L^2(0,d)} - \eta^2\|R'\|^2_{L^2(0,d)}$$

$$- 2\eta\chi_1'(0)\int_{aM}\phi_1^{(a)}(x)\,\mathrm{d}x \tag{3.5}$$

where the negative term in the bracket can, of course, be neglected. The second and the third terms on the right-hand side can be estimated in analogy with [EV1]:

$$\mu_1(a)\|R\|^2_{L^2(0,d)} - \eta^2\|R'\|^2_{L^2(0,d)} < \frac{\sqrt{\mu_1(1)}}{2a}(2+\varepsilon_1)$$

for fixed $\varepsilon_1 > 0$ and any a sufficiently small. In a similar way, the last term equals $-2\eta\chi_1'(0)Ca$, where $C := \int_M \phi_1^{(1)}(x)\,\mathrm{d}x$. Finally, the first term can be evaluated by means of [AS, equation 9.6.26] and [PBM, equation 1.12.3.2]:

$$K_0(\kappa a)^2\|\nabla f_1\|^2_{L^2(\mathbb{R}^2)} = 2\pi\left[\tfrac{1}{2}\kappa^2a^2K_1'(\kappa a)^2 - \tfrac{1}{2}(\kappa^2a^2+1)K_1(\kappa a)^2\right].$$

Using $-K_1'(\xi) = K_0(\xi) + \xi^{-1}K_1(\xi)$ in combination with the asymptotic expressions $K_0(\xi) = -\ln\xi + \mathcal{O}(1)$, $K_1(\xi) = \xi^{-1} + \mathcal{O}(\ln\xi)$, we find

$$\|\nabla f_1\|^2_{L^2(\mathbb{R}^2)} < -\frac{2\pi(1+\varepsilon_2)}{\ln\kappa a}$$

for a fixed ε_2 and a sufficiently small. Substituting these estimates in (3.5) and taking a minimum over η, we arrive at the bound

$$L(\psi) < -\frac{2\pi(1+\varepsilon_2)}{\ln\kappa a} - \frac{2\chi_1'(0)^2C^2}{(2+\varepsilon_1)\sqrt{\mu_1(1)}}a^3. \tag{3.6}$$

It remains for us to find a lower bound on

$$\|\psi\|^2 \geqslant \|\psi\|^2_{|x|\geqslant a} - 2\|F\|^2_{|x|\leqslant a} - 2\eta^2\|F\|^2_{|x|\leqslant a} = \|\psi\|^2_{|x|\geqslant a} - 2\pi a^2 - 2\eta^2\|R\|^2_{L^2(0,d)}.$$

The last term is $\mathcal{O}(a)$, while the first can be expressed as

$$K_0(\kappa a)^2\|F\|^2_{|x|\geqslant a} = \pi a^2\left[K_1(\kappa a)^2 - K_0(\kappa a)^2\right] = \frac{\pi}{\kappa^2} + \mathcal{O}(a^2\ln\kappa a).$$

Using the asymptotic behaviour of K_0 we find $\|\psi\|^2 \geqslant \pi\kappa^{-2}(\ln\kappa a)^{-2}(1-\varepsilon_3)$ for fixed $\varepsilon_3 > 0$ and sufficiently small a. Hence

$$\frac{L(\psi)}{\|\psi\|^2} < -\frac{\kappa^2\ln\kappa a}{\pi(1-\varepsilon_3)}(Da^3\ln\kappa a + E) \tag{3.7}$$

where $E := 2\pi(1+\varepsilon_2)$ and

$$D := \frac{2\chi_1'(0)^2C^2}{(2+\varepsilon_1)\sqrt{\mu_1(1)}}.$$

Minimizing the right-hand side of (3.7) with respect to κ, we conclude that for fixed positive ε_1, ε_2 and $\varepsilon_3 \in (0,1)$ there is a function g such that

$$\frac{L(\psi)}{\|\psi\|^2} < g(a) \qquad \text{and} \qquad g(a) \approx -\frac{1+\varepsilon_2}{1-\varepsilon_3}\,\frac{1}{a^2}\,\mathrm{e}^{-2E/Da^3} \tag{3.8}$$

as $a \to 0$. The upper bound in (3.1) follows readily from (3.8); the extension to the non-symmetric case is obtained as in [EV1].

Remark 3.2. In fact, one could suppose $M = B_1$ because the eigenvalue is pushed up if we reduce the window to a circle contained in M, and the bound obtained is non-optimal, as in remark 2.2. In the rest of the proof we *embed* M in a circle, leaving the question about relations between the constants and the geometry of M to more sophisticated methods.

The lower bound can again only be proved in the symmetric case. We begin with auxiliary results. When constructing the trial function component (3.2), we have implicitly used the fact that the functional $F:\ F(\phi) = \int_a^\infty \big(\phi'(t)^2 + m^2\phi(t)^2\big)t\,\mathrm{d}t$ on $C^2([a,\infty))$ with the condition $\phi(a) = \alpha$ and fixed positive a, m and is minimized by

$$\phi_0:\ \phi_0(t) = \alpha\frac{K_0(mt)}{K_0(ma)} \tag{3.9}$$

as can easily be seen from solution of the appropriate Euler equation. Furthermore, a two-dimensional analogy of the bound [EV1, equation (4.2)] is given by the *Friedrichs inequality* [Ne, theorem 1.9]: if $\Omega \subset \mathbb{R}^n,\ n \geqslant 2$, is a bounded domain with Lipschitz boundary, there is a positive c such that

$$\|\nabla f\|^2 \geqslant c\|f\|^2 \tag{3.10}$$

holds for every $f \in H_0^1(\Omega)$. The constant is, of course, easy to find for the circle $\Omega = B_a$ in terms of the appropriate Bessel zero, $c = j_{0,1}^2 a^{-2}$.

Repeating the argument of [EV1] and the previous section, we infer that one has to find a lower bound on $L(\psi)/\|\psi\|^2$ over all real $\psi \in L^2(\Sigma)$, which are C^2, radially symmetric, and vanish at the boundary except in the window. We can again express such a ψ in the form of the series (2.12), where the convergence is uniform for $|x| \geqslant a$. The coefficients c_n in fact depend only on $r := |x|$. Moreover, in analogy with (2.22) we may restrict our attention to trial functions with

$$|c_n(r)| \leqslant |c_n(a)|\frac{K_0\big((\pi/d)\sqrt{n^2-1}r\big)}{K_0\big((\pi/d)\sqrt{n^2-1}a\big)} \tag{3.11}$$

for $n \geqslant 2$. As before we introduce

$$F(x,y) := \begin{cases} \alpha\chi_1(y) & 0 \leqslant r \leqslant 2a \\ c_1(r)\chi_1(y) & r \geqslant 2a \end{cases} \tag{3.12}$$

with $\alpha := c_1(2a)$, and divide the rest $G(x,y) = \psi(x,y) - F(x,y)$ into

$$G_1(x,y) := (c_1(r) - \alpha)\chi_1(y)$$

supported in the extended window region, $r \leqslant 2a$, and $G_2(x,y) = \hat{G}(x,y) + \Gamma(x,y)$ with

$$\Gamma(x,y) := \sum_{n=2}^\infty c_n(2a)\chi_n(y).$$

We start estimating the reduced energy functional

$$L(\psi) = \|\nabla_x\psi\|^2 + \|G_y\|^2 - \left(\frac{\pi}{d}\right)^2\|G\|^2 - 2\alpha\chi_1'(0)\int_{B_a} G(x,0)\,\mathrm{d}x \tag{3.13}$$

from the 'external' contribution to the first 'two-and-a-half' terms:

$$\begin{aligned} L_1 &:= \frac{1}{2}\|\nabla_x\psi\|_{r\geqslant a}^2 + \|G_y\|_{r\geqslant a}^2 - \left(\frac{\pi}{d}\right)^2\|G\|_{r\geqslant a}^2 \\ &= \pi\sum_{n=1}^\infty\int_a^\infty\left(c_n'(r)^2 + 2\left(\frac{\pi}{d}\right)^2(n^2-1)c_n(r)^2\right)r\,\mathrm{d}r \end{aligned}$$

$$\geqslant \pi \sum_{n=2}^{\infty} \int_a^{\infty} \left(c_n'(r)^2 + 2\left(\frac{\pi n}{d}\right)^2 c_n(r)^2 \right) r\,\mathrm{d}r$$

$$\geqslant \pi \sum_{n=2}^{\infty} c_n(a)^2 \frac{\pi n}{d} a \frac{K_1(\pi n a/d)}{K_0(\pi n a/d)} \geqslant \frac{\pi^2 a}{d} \sum_{n=2}^{\infty} n c_n(a)^2 \tag{3.14}$$

where in the last line we have used equation (3.9), evaluated the integral as in the first part of the proof, and employed the inequality $K_1(\xi) \geqslant K_0(\xi)$ which follows from the well known integral representation [AS, equation 9.6.24]. Next we turn to

$$L_2 := \|\nabla_x \psi\|^2_{r\leqslant 2a} = \|\nabla_x G_1\|^2_{r\leqslant 2a} + \|\nabla_x G_2\|^2_{r\leqslant 2a}. \tag{3.15}$$

By assumption, G_1 vanishes at $r = 2a$, so the first term can be estimated from (3.10) as

$$\|\nabla_x G_1\|^2_{r\leqslant 2a} \geqslant \frac{C_1}{4a^2} \|G_1\|^2_{r\leqslant 2a} = \frac{C_1}{a^2} \|G_1\|^2 \tag{3.16}$$

where $4C_1 := j_{0,1}^2$. Furthermore, introducing the window neighbourhood $\Omega_a := B_{2a} \times [0, a]$, we have

$$\|\nabla_x G_2\|^2_{r\leqslant 2a} = \|\nabla_x \hat{G}\|^2_{r\leqslant 2a} \geqslant \frac{C_1}{a^2} \|\hat{G}\|^2_{r\leqslant 2a}$$

$$\geqslant \frac{C_1}{a^2} \|\hat{G}\|^2_{\Omega_{2a}} \geqslant \frac{\delta C_1}{a^2} \|\hat{G}\|^2_{\Omega_{2a}} \geqslant \frac{\delta C_1}{2a^2} \|G_2\|^2_{\Omega_{2a}} - \frac{\delta C_1}{a^2} \|\Gamma\|^2_{\Omega_{2a}} \tag{3.17}$$

for all $a \leqslant d$ and $\delta \in (0, 1]$. The last norm can be estimated as in the previous cases by combining the smallness of the χ_n norm restricted to $[0, a]$ with the dominated decay (3.11):

$$\|\Gamma\|^2_{\Omega_a} = 4\pi a^2 \int_0^a \left(\sum_{n=2}^{\infty} c_n(2a) \chi_n(y) \right)^2 \mathrm{d}y$$

$$\leqslant 8\pi a^2 \left(\sum_{n=2}^{[a^{-1}]+1} n^{-1} c_n(a)^2 \int_0^a \chi_n(y)^2\, \mathrm{d}y \right) \sum_{n=2}^{[a^{-1}]+1} n$$

$$+ 8\pi a^2 \left(\sum_{2\leqslant n=[a^{-1}]+2}^{\infty} n c_n(a)^2 \int_0^a \chi_n(y)^2\, \mathrm{d}y \right)$$

$$\times \sum_{2\leqslant n=[a^{-1}]+2}^{\infty} \frac{K_0^2\big((2\pi a/d)\sqrt{n^2-1}\big)}{n K_0^2\big((\pi a/d)\sqrt{n^2-1}\big)}$$

$$\leqslant \frac{16\pi a^3}{d} \left(\frac{2\pi^2}{3d^2} + \sum_{2\leqslant n=[a^{-1}]+2}^{\infty} \frac{K_0^2\big((2\pi a/d)\sqrt{n^2-1}\big)}{n K_0^2\big((\pi a/d)\sqrt{n^2-1}\big)} \right) \sum_{n=2}^{\infty} n c_n(a)^2.$$

The sum in the bracket can be estimated as

$$\sum_{2\leqslant n=[a^{-1}]+2}^{\infty} \frac{K_0^2\big((2\pi a/d)\sqrt{n^2-1}\big)}{n K_0^2\big((\pi a/d)\sqrt{n^2-1}\big)} \leqslant \int_{a^{-1}}^{\infty} \frac{K_0^2\big((2\pi a/d)\sqrt{n^2-1}\big)}{\xi K_0^2\big((\pi a/d)\sqrt{n^2-1}\big)}\, \mathrm{d}\xi \leqslant \int_1^{\infty} \frac{K_0^2(\pi\xi/d)}{\xi K_0^2(\pi\xi/2d)}$$

for $a < \sqrt{3}/2$, and the integral on the right-hand side is convergent, because $K_0(\xi) \approx \sqrt{\pi/2\xi}\, \mathrm{e}^{-\xi}$ as $\xi \to \infty$. Hence there is a positive C_2 independent of ψ and a such that

$$\frac{C_1}{a^2} \|\Gamma\|^2_{\Omega_a} < C_2 a \sum_{n=2}^{\infty} n c_n(a)^2. \tag{3.18}$$

Combining the estimates (3.14)–(3.18), we arrive at

$$L_1 + L_2 \geqslant a\left(\frac{\pi^2}{d} - \delta C_2\right)\sum_{n=2}^{\infty} n c_n(a)^2 + \frac{C_1}{a^2}\|G_1\|^2 + \frac{\delta C_1}{2a^2}\|G_2\|^2_{\Omega_a}$$

which gives

$$L_1 + L_2 \geqslant \frac{C_1}{a^2}\|G_1\|^2 + \frac{m^2}{a^2}\|G_2\|^2_{\Omega_a} \tag{3.19}$$

for some $m > 0$ and all a sufficiently small.

The norm of G_y is estimated as in the two-dimensional case [EV1]:

$$\|G_y\|^2_{r\leqslant a} \geqslant \|G_{2,y}\|^2_{r\leqslant a} - \frac{2\pi}{d^2}\left(2\|G_1\|^2_{r\leqslant a} + d\|G_2(\cdot,0)\|^2_{r\leqslant a}\right)$$

which, together with (3.19), yields

$$\begin{aligned}
L_1 + L_2 + \|G_y\|^2_{r\leqslant a} - \left(\frac{\pi}{d}\right)^2\|G\|^2_{r\leqslant a}& \\
&\geqslant \|G_{2,y}\|^2_{r\leqslant a} - \left(\frac{\pi}{d}\right)^2\|G_2\|^2_{r\leqslant a} + \left(\frac{C_1}{a^2} - \frac{\pi(\pi+4)}{d^2}\right)\|G_1\|^2_{r\leqslant a}\\
&\quad + \frac{m^2}{a^2}\|G_2\|^2_{\Omega_a} - \frac{2\pi}{d}\|G_2(\cdot,0)\|^2_{r\leqslant a}\\
&\geqslant \left(\frac{c_0}{a} - \frac{2\pi}{d}\right)\|G_2(\cdot,0)\|^2_{r\leqslant a}\\
&\geqslant \frac{c_0}{2a}\|G_2(\cdot,0)\|^2_{r\leqslant a}
\end{aligned}$$

for positive c_0 and any a sufficiently small; in the second step we have neglected a positive term and employed lemma 2.5. Substituting from here to (3.13) and using the Schwarz inequality

$$\int_{B_a} G(x,0)\,\mathrm{d}x \leqslant \|G_2(\cdot,0)\|_{r\leqslant a}\sqrt{\pi}\,a$$

we obtain

$$\begin{aligned}
L(\psi) &\geqslant \frac{1}{2}\|\nabla_x\psi\|^2_{r\geqslant 2a} - 2\alpha a\chi_1'(0)\sqrt{\pi}\|G_2(\cdot,0)\|_{r\leqslant a} + \frac{c_0}{2a}\|G_2(\cdot,0)\|^2_{r\leqslant a}\\
&\geqslant \frac{1}{2}\|\nabla_x\psi\|^2_{r\geqslant 2a} - \frac{2\pi\alpha^2\chi_1'(0)^2}{c_0}a^3.
\end{aligned}$$

The first term on the right-hand side can be estimated from below by the first transverse-mode contribution. The same applies to $\|\psi\|^2$, so finally we find

$$\frac{L(\psi)}{\|\psi\|^2} \geqslant \frac{\int_{2a}^{\infty} c_1'(r)^2\, r\mathrm{d}r - (\pi\chi_1'(0)^2/c_0)a^3 c_1(2a)^2}{2\int_{2a}^{\infty} c_1(r)^2\, r\mathrm{d}r}. \tag{3.20}$$

In analogy with (3.9), one has to solve the appropriate Euler equation to check that the right-hand side of (3.20) is minimized by $c_1 = \phi_\kappa$ for some $\kappa > 0$, where

$$\phi_\kappa(r) := c_1(2a)\frac{K_0(\kappa r)}{K_0(2\kappa a)}.$$

Substituting in (3.20), evaluating the integrals, and taking the asymptotics for small a, we infer that

$$\frac{L(\psi)}{\|\psi\|^2} \geqslant -\kappa^2 \ln(2\kappa a)\left(\frac{\pi\chi_1'(0)^2}{c_0(1+\varepsilon_2)}a^3\ln(2\kappa a)+\frac{1-\varepsilon_1}{1+\varepsilon_2}\right)$$

holds for any fixed $\varepsilon_1, \varepsilon_2 > 0$ and all sufficiently small a. It remains to find the minimum of the right-hand side with respect to κ. However, since it differs from (3.7) just by the values of the constants, the argument is concluded as in the first part of the proof. □

4. Conclusions

To make sense of the bounds derived in the above sections one has to take into account two aspects of the problem. First of all, we have already mentioned that the discrete spectrum can also be found numerically by means of the mode-matching method; a detailed description of the two-dimensional case is given in [EŠTV]. Although the method converges rather slowly if the window is narrow, the results obtained for a single window clearly suggest that the true asymptotics exist and are of the same type as our asymptotic bounds.

Further insight can be obtained from comparing our result with the well known weak-coupling asymptotics for Schrödinger operators in dimension one and two [BGS, Kl, Si]. The ground state of the coupled strips in the narrow-window case is dominated the lowest transverse-mode component with long exponentially decaying tails and a local modification in the coupling region. In a similar way, a link can be made between window-connected layers and a two-dimensional Schrödinger operator. The comparison shows that the attractive interaction due to opening a narrow window (in particular, by changing the Dirichlet boundary condition to Neumann over a short segment of the boundary in the symmetric case) acts effectively as a potential well of a depth proportional to the size of the window.

Conjecture 4.1. Let $H(d_1, d_2; \mathcal{W})$ be the operators described above. The ground-state eigenvalue behaves for small $|\mathcal{W}|$ as

$$\epsilon(a) \approx \left(\frac{\pi}{d}\right)^2 - \frac{1}{d^2}\left(\sum_{k=1}^{N} c_{2,k}(\nu)a_k^2\right)^2 \qquad \dim\Sigma = 2 \tag{4.1}$$

$$\epsilon(a) \approx \left(\frac{\pi}{d}\right)^2 - \frac{1}{d^2}\exp\left\{-\left(\sum_{k=1}^{N} c_{3,k}(\nu)a_k^3\right)^{-1}\right\} \qquad \dim\Sigma = 3 \tag{4.2}$$

where $\nu := d^{-1}\min\{d_1, d_2\}$, and a_k in the three-dimensional case is the scaling parameter of the kth window.

The conjecture is based solely on the analogy described, and therefore it is difficult to say more about the coefficients. The possibility that they may depend on the geometry of the window-centre set for $N > 1$ is not excluded; in the three-dimensional case the shapes of the scaled windows may also play a role. We refrain from speculating about the nature of the error terms.

On the other hand, we are convinced that the open 'constant cross-section' shape of our regions Σ is crucial for the asymptotics. For instance, if Σ is instead a bounded planar region with the Dirichlet boundary in which we open a window (to another bounded region the essential spectrum threshold of which is not lower) or a Neumann segment, we conjecture that leading term in the ground-state shift is proportional to the *square* of the window width. Moreover, the same asymptotics are expected to be valid for higher eigenvalues provided the corresponding eigenfunctions are locally *symmetric* with respect to the window axis. In any case, proving such asymptotic properties represents an intriguing mathematical problem.

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